Critical manifold of globally coupled overdamped anharmonic oscillators driven by additive Gaussian white noise

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We consider an infinite array of globally coupled overdamped anharmonic oscillators subject to additive Gaussian white noise which is closely related to the mean field Φ^4-Ginzburg-Landau model. We prove the existence of a well-behaved critical manifold in the parameter space which separates a symmetric phase from a symmetry broken phase. Given two of the system parameters, there is a unique critical value of the third. The proof exploits that the critical control parameter a_c is bounded by its limit values for weak and strong noise. In these limits, the mechanism of symmetry breaking differs. For weak noise, the distribution is Gaussian and the symmetry is broken as the whole distribution is shifted in either the positive or the negative direction. For strong noise, there is a symmetric double-peak distribution and the symmetry is broken as the weights of the peaks become different. We derive an ordinary differential equation whose solution describes the critical manifold. Using a series ansatz to solve this differential equation, we determine the critical manifold for weak and strong noise and compare it to numerical results. We derive analytic expressions for the order parameter and the susceptibility close to the critical manifold.

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I. INTRODUCTION

Nonlinear globally coupled systems under the influence of noise have been an active field of research over the last few decades [1,2]. For additive noise, there is a natural and far-reaching analogy to equilibrium thermodynamics [3].

We consider an array of L harmonically coupled overdamped anharmonic oscillators subject to additive noise, which is governed by the system of Langevin equations

$$\dot{x}_i = a x_i - x_i^3 - \frac{D}{L-1}\sum_{j=1}^{L}(x_i - x_j) + \xi_i(t) \qquad (1)$$

for $i = 1,\dots,L$. Each of the oscillators is harmonically coupled to all the others (global coupling), and the total strength of the coupling is D. The additive noise $\xi_i(t)$ is a zero mean Gaussian white process with autocorrelation

$$\langle \xi_i(t)\xi_j(s)\rangle = \sigma^2 \delta_{ij}\delta(t-s), \qquad (2)$$

where σ^2 denotes the noise strength. The steady state of an isolated system without noise ($D = 0$, $\sigma = 0$) undergoes a pitchfork bifurcation if the control parameter a changes the sign. The isolated system ($D = 0$) describes diffusion in a potential with one or two minima depending on the sign of a which is thoroughly studied since [4]. In the limit $L \to \infty$, the array exhibits a continuous phase transition accompanied with a symmetry breaking above the critical point $a_c(D,\sigma)$.

The dependence of a_c on the other parameters has been investigated already by a number of authors [5–9]. Kometani and Shimizu [5] closed the equation of motion for the moments using a decoupling which is correct for Gaussian distributed variables. The critical point is determined by the occurrence of a nontrivial solution. This yields, in our notation, $a_c = 3\sigma^2/(2D)$ which is asymptotically correct for weak noise (see the following). Desai and Zwanzig [6] described the system by a self-consistent dynamic mean field theory. They evaluated numerically the correct phase transition condition and observed that the critical point deviates from the result of a Gaussian approximation. Dawson [7] correctly claimed existence and uniqueness of the critical parameter. We show that he used a wrong argument, and we give a different proof of uniqueness. He proved in the limit of infinitely many oscillators that the fluctuations at the critical point are non-Gaussian and occur at a slower time scale than the noncritical fluctuations. Furthermore, he observed that for $D = a$ the critical point can be computed up to a quadrature. Van den Broeck and collaborators [8] also gave numerical results for the parameter dependence of a_c in mean field theory and compared this with simulations for a system with nearest neighbor coupling in $d = 2$. Implicitly, they state that for strong coupling ($D \to \infty$), the critical point is $a_c = 0$ (cf. also [9]).

Shiino [10,11] proved an H-theorem showing thus that the stationary state is asymptotically reached for long times and analyzed the stability of the trivial and the symmetry breaking solutions.

The harmonic coupling between the constituents of an array was introduced by Kometani and Shimizu [5] to describe the interaction between myosin and actin filaments in muscle contraction. In this context, the variables x_i represent velocities rather than coordinates and the system can be regarded as an early example of a canonical-dissipative system [12], which has been proposed to describe, e.g., swarm dynamics [13].

The harmonic coupling between nearest neighbors of a regular lattice can be conceived as discretization of the Laplace operator Δ and is therefore also called diffusive coupling. Thus, in the continuum limit there is a relation to mean field solutions of a class of models described by stochastic partial differential equations (see, e.g., [14]).

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There is further a relation to the discretized version of the Φ^4-Ginzburg-Landau model, in different context also known as soft-spin Ising model. For example, in [15] the authors studied spin glasses, where the coupling strength for each pair of coordinates x_i, x_j is an independent Gaussian distributed random variable.

Noise induced phenomena in a double-well potential are still a topic of recent research. For example, in [16] the authors investigated a Fokker-Planck equation driven by dynamical constraints, modeling many-particle storage systems.

For a similar system driven by multiplicative noise instead of additive noise, also continuous phase transitions occur. The critical exponent of the order parameter $\langle x\rangle$ undergoes a transition from a constant (although non-mean-field) value towards a parameter-dependent value when $\sigma^2/(2D)$ exceeds a threshold [17]. Also, higher moments $\langle x^m\rangle$ show such transitions [18,19]. It is a natural question as to whether this behavior is robust against additive noise since in natural systems additive noise is apparently unavoidable. Although many papers study systems with additive and multiplicative noise (for early references, see [8,9]), this question has not been explicitly addressed. To determine critical exponents, it is advantageous to know analytically the parameter dependence of the critical manifold. Aside from our general interest, this is an additional motivation to study the present system.

The paper is organized as follows. In Sec. II, we reformulate the model in the Fokker-Planck picture and explain the self-consistent mean field approach which becomes exact in the limit of infinite system size. We show that the self-consistency condition is equivalent with the stationarity condition for the center of mass variable. In Sec. III, we prove the existence of a well-behaved critical manifold in the parameter space which separates the regime with $\langle x\rangle = 0$ from the regime with broken symmetry. The critical manifold is determined by an implicit integral equation, the phase transition condition (PTC). To show its well-behavedness, it is necessary to know that the critical parameter $a_c(D,\sigma)$ is bounded by values which are asymptotically reached in the limits of strong and weak noise, respectively. The well-behavedness of the critical manifold allows us to reduce the number of parameters by rescaling and implies that a_c/D is only a function of the ratio σ/D. Thus, it exhibits the same behavior for strong coupling as for weak noise, and for weak coupling as for strong noise.

Exploiting that the critical manifold is well behaved, together with exact relations of moments on this manifold we derive an ordinary differential equation (ODE) for a_c/D as a function of $(\sigma/D)^2$. To determine a solution for small and large values of $(\sigma/D)^2$, we use a series expansion. The results from the ODE are the same as from an evaluation of the PTC using the Laplace method [20] but easier to obtain. The asymptotic behavior of the series agrees with that of the numerical solution of the PTC. We were not able to prove convergence of these series, but for small $(\sigma/D)^2$ a Padé approximant agrees well with the numerical solution. We also give results for a series expansion around $a = D$ which is analytically treatable.

In Sec. IV, we determine the behavior of the order parameter and the susceptibility near the critical manifold. Thanks to the boundedness of a_c, the existence of a tricritical point is excluded and the critical exponents are the mean field exponents, as expected. We derive the amplitudes of the power laws of order parameter and susceptibility in closed form in terms of a_c. The amplitude ratio of the susceptibilities just above and below the critical point is universal as expected by analogy with equilibrium second order phase transitions.

Several detailed calculations and technical discussions are deferred to the appendixes. In Appendix A, recursion relations for the moments are given. In Appendix B, we derive in some detail the bounds of a_c.

II. FOKKER-PLANCK PICTURE

The Fokker-Planck equation corresponding to Eq. (1) is

$$\partial_t p(\mathbf{x},t) = \sum_{i=1}^{L} -\partial_{x_i}\left\{\left[(a-D)x_i - x_i^3 + \frac{D}{L-1}\sum_{j=1}^{L} x_j - \frac{\sigma^2}{2}\partial_{x_i}\right] p(\mathbf{x},t)\right\}. \tag{3}$$

Integrating over all coordinates but x_1 we find

$$\partial_t p_1^L(x_1,t) = -\partial_{x_1}\left\{\left[(a-D)x_1 - x_1^3 + D\langle x_2|x_1\rangle - \frac{\sigma^2}{2}\partial_{x_1}\right] p_1^L(x_1,t)\right\}, \tag{4}$$

where $\langle x_2|x_1\rangle$ is the conditional expectation value of x_2 given x_1 and $p_1^L(x_1,t)$ is the probability distribution of x_1 for the system with L constituents. For $L\to\infty$, we assume independence of coordinates, i.e.,

$$\langle x_2|x_1\rangle \approx \langle x_2\rangle = \langle x_1\rangle \tag{5}$$

and find the one-particle distribution as the solution of

$$\partial_t p(x,t) = -\partial_x\left\{\left[(a-D)x - x^3 + D\int_{-\infty}^{\infty} dx' x' p(x',t) - \frac{\sigma^2}{2}\partial_x\right] p(x,t)\right\}, \tag{6}$$

where we wrote $p(x,t)$ instead of $p_1^{L\to\infty}(x_1,t)$.

In [7], a more rigorous approach is proposed. Provided that for $t = 0$ all coordinates are independent and identically distributed with distribution $p_0(x)$, one can show that in the limit $L\to\infty$ the empirical distribution of all constituents converges weakly to $p(x,t)$ which is the unique solution of Eq. (6) with initial condition $p(x,t=0) = p_0(x)$. Hence, in the limit $L\to\infty$, the mean field description given by Eq. (6) becomes exact.

For an infinitely large system, the harmonic coupling in Eq. (1) of the site i to all other sites becomes

$$-\frac{D}{L-1}\sum_{j=1}^{L}(x_i - x_j) \to -D(x_i - m), \tag{7}$$

where

$$m(t) = \lim_{L\to\infty}\frac{1}{L}\sum_{j=1,j\neq i}^{L} x_j = \lim_{L\to\infty}\int_{-\infty}^{\infty} dx\, x p_1^L(x,t) = \int_{-\infty}^{\infty} dx\, x p(x,t) \tag{8}$$

is the mean field exerted by all other sites to the system at site i.

It was shown in [10,11] that the time dependent solution of Eq. (6) approaches the stationary solution p_s for long times. In the stationary case, m is constant and can be considered as a parameter. We find the stationary solution of Eq. (6):

$$p_s(x;m) = \frac{1}{Z}\exp\left[\frac{2}{\sigma^2}\left((a-D)\frac{x^2}{2} - \frac{x^4}{4} + Dmx\right)\right], \quad (9)$$

with normalization

$$Z = \int_{-\infty}^{\infty} dx \exp\left[\frac{2}{\sigma^2}\left((a-D)\frac{x^2}{2} - \frac{x^4}{4} + Dmx\right)\right], \quad (10)$$

satisfying

$$p_s(x;m) = p_s(-x;-m). \quad (11)$$

The mean field should solve the self-consistency equation

$$m = \int_{-\infty}^{\infty} dx\, x p_s(x;m). \quad (12)$$

Obviously, the mean field is related to the center of mass coordinate

$$R = \frac{1}{L}\sum_{i=1}^{L} x_i, \quad (13)$$

obeying the Langevin equation

$$\dot{R} = \frac{1}{L}\sum_{i=1}^{L}\left(ax_i - x_i^3\right) + \frac{1}{L}\sum_{i=1}^{L}\xi_i(t). \quad (14)$$

For the infinite system, the noise term vanishes due to the law of large numbers and Eq. (14) becomes

$$\lim_{L\to\infty}\dot{R} = a\langle x\rangle - \langle x^3\rangle. \quad (15)$$

In the stationary case, $a\langle x\rangle - \langle x^3\rangle = 0$. This is equivalent to the self-consistency condition (12) as can be seen writing

$$\begin{aligned}
&\int_{-\infty}^{\infty} dx\,(ax - x^3)p_s(x;m)\\
&= \frac{1}{Z}\frac{\sigma^2}{2}\int_{-\infty}^{\infty} dx\left\{\partial_x \exp\left[\frac{2}{\sigma^2}\left(\frac{a}{2}x^2 - \frac{1}{4}x^4\right)\right]\right\}\\
&\quad\times \exp\left[\frac{2D}{\sigma^2}\left(-\frac{1}{2}x^2 + mx\right)\right]\\
&= D(\langle x\rangle - m) = 0. \quad (16)
\end{aligned}$$

The second equality follows by partial integration and observing that the boundary term vanishes since $p_s(x;m)$ decays exponentially fast for $|x| \to \infty$.

III. CRITICAL MANIFOLD

A. Existence and general properties

We introduce the function

$$F(m) := \int_{-\infty}^{\infty} dx\, x p_s(x,m), \quad (17)$$

satisfying the symmetry

$$F(m) = -F(-m). \quad (18)$$

The self-consistency equation (12) now reads as

$$m = F(m), \quad (19)$$

which has an obvious solution $m = 0$. If a nonzero solution m_+ exists, then $m_- = -m_+$ is also a solution of Eq. (19). For $m = 0$, the stationary distribution $p_s(x;0)$ is symmetric with respect to $x \to -x$, whereas this symmetry is broken for a nonzero solution [cf. Eq. (9)]. For $m > 0$, the curvature of $F(m)$ is negative as shown in [7] using a simple version of the Griffiths-Hurst-Sherman inequality [21]. If the derivative of F at $m = 0$ is larger than one, there exists exactly one positive solution to Eq. (19) [7]. Then, by symmetry we also have a negative solution. Otherwise, $m = 0$ is the only solution. Shiino [10,11] showed that the solutions with $m \neq 0$ are stable if they exist whereas the $m = 0$ solution is unstable in that case. The phase transition condition

$$\partial_m F(m, a = a_c)|_{m=0} = 1 \quad (20)$$

can be written as

$$\phi(a,D,\sigma) = 0, \quad (21)$$

where

$$\phi(a,D,\sigma) := \frac{\frac{2D}{\sigma^2}\int_{-\infty}^{\infty} dx\, x^2 \exp\left[\frac{2}{\sigma^2}\left(\frac{a-D}{2}x^2 - \frac{1}{4}x^4\right)\right]}{\int_{-\infty}^{\infty} dx\, \exp\left[\frac{2}{\sigma^2}\left(\frac{a-D}{2}x^2 - \frac{1}{4}x^4\right)\right]} - 1. \quad (22)$$

This defines the critical manifold in the space spanned by (a,D,σ). We observe immediately that on the critical manifold we have

$$\langle x^2\rangle|_{\text{crit}} = \frac{\sigma^2}{2D}. \quad (23)$$

In the following, we show that the critical manifold is well behaved: given any two of the parameters a, D, or σ, there exists a unique value, the critical value, of the third parameter which solves Eq. (21). For $D \leqslant 0$, there is no solution to (21). Therefore, we consider $D > 0$ and furthermore $\sigma > 0$ since there are only contributions in σ^2, negative σ is equivalent. ϕ is continuous and continuously differentiable in a, D, and σ. ϕ is even C^∞ on $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$. By asymptotic evaluation of the integrals in (22), we find

$$\lim_{a\to\infty}\phi(a,D,\sigma) = +\infty, \quad (24)$$

$$\lim_{a\to-\infty}\phi(a,D,\sigma) = -1. \quad (25)$$

Because of continuity, for every $D,\sigma > 0$ there exists an a satisfying Eq. (21). Since

$$\partial_a\phi(a,D,\sigma) = \frac{2D}{\sigma^4}(\langle x^4\rangle - \langle x^2\rangle^2) > 0, \quad (26)$$

this solution is unique, $a = a_c$. The critical parameter a_c is bounded by

$$\frac{1}{2}\frac{\sigma^2}{D} < a_c < \frac{3}{2}\frac{\sigma^2}{D}, \quad (27)$$

as proven in Appendix B. These are the best possible bounds since the upper and the lower bounds are asymptotically reached for weak or strong noise, respectively, as shown in Secs. III C and III D.

Since $a_c > 0$, we consider ϕ on $\mathbb{R}_+^3$. Because of Eq. (26), we can apply the implicit function theorem: there is locally around a solution of Eq. (21) a unique C^∞ function $f_a(D,\sigma) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\phi(f_a(D,\sigma),D,\sigma) = 0. \tag{28}$$

Since there exists a unique solution $a_c = f_a(D,\sigma)$ to Eq. (21) for every $D,\sigma > 0$, the function $f_a(D,\sigma)$ is globally uniquely defined on $\mathbb{R}_+^2$ and is in C^∞. Looking at relation (27), we see furthermore that $a_c = f_a(D,\sigma)$ assumes all values in $\mathbb{R}_+$ even if one of the variables D or σ is fixed to some value. We consider a fixed $\sigma = \sigma_0$ and exploit that $\phi(f_a(D,\sigma_0),D,\sigma_0) = 0$ for all D. Therefore,

$$\begin{aligned} 0 &= \frac{d}{dD}\phi(f_a(D,\sigma_0),D,\sigma_0) \\ &= \frac{2}{\sigma_0^2}\langle x^2\rangle\Big|_{a_c} - \frac{2D}{\sigma_0^4}(\langle x^4\rangle|_{a_c} - \langle x^2\rangle^2|_{a_c})\left(1 - \frac{\partial f_a}{\partial D}\right). \end{aligned} \tag{29}$$

Solving for $\partial_D f_a$ and inserting the relations (23) and (A4), we find

$$\frac{\partial}{\partial D} f_a(D,\sigma_0) = \frac{a_c - \frac{3\sigma_0^2}{2D}}{a_c - \frac{\sigma_0^2}{2D}} < 0, \tag{30}$$

where the negativity is guaranteed by inequality (27). Because of the monotonicity and surjectivity of $f_a(D,\sigma_0)$, there exists an inverse function $f_D(a,\sigma_0) : \mathbb{R}_+ \to \mathbb{R}_+$ for all $\sigma_0 > 0$, i.e., we have the function $f_D(a,\sigma) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\phi(a, f_D(a,\sigma),\sigma) = 0. \tag{31}$$

Analogously, for fixed $D = D_0$, we calculate

$$\begin{aligned} 0 &= \frac{d}{d(1/\sigma^2)}\phi(f_a(D_0,\sigma),D_0,\sigma) \\ &= 2D_0\langle x^2\rangle + \frac{2D_0}{\sigma^2}\left[(f_a - D_0)\langle x^4\rangle - \frac{1}{2}\langle x^6\rangle\right] \\ &\quad - \frac{2D_0}{\sigma^2}\left[(f_a - D_0)\langle x^2\rangle^2 - \frac{1}{2}\langle x^2\rangle\langle x^4\rangle\right] \\ &\quad + \frac{2D_0}{\sigma^4}(\langle x^4\rangle - \langle x^2\rangle^2)\frac{\partial f_a}{\partial(1/\sigma^2)}. \end{aligned} \tag{32}$$

Inserting the expressions (23), (A4), and (A5) we find

$$\frac{\partial f_a}{\partial(1/\sigma^2)} = -\frac{\sigma^2}{2}\left(D_0\frac{\frac{3\sigma^2}{2D_0} - a_c}{a_c - \frac{\sigma^2}{2D_0}} + a_c\right) < 0, \tag{33}$$

where inequality (27) was used. Hence, we find

$$\frac{\partial}{\partial\sigma} f_a(D,\sigma) > 0. \tag{34}$$

From the monotonicity there follows the existence of a function $f_\sigma(a,D)$ satisfying

$$\phi(a,D,f_\sigma(a,D)) = 0 \tag{35}$$

in analogy to the previous case. Hence, given any two parameters, positive, there exists a unique critical value of the third parameter denoted by f_a, f_D, or f_σ. Because of the monotonicity of ϕ with respect to a and of f_a with respect to D and σ we conclude that there exists a pair of nonzero solutions $m_\pm$ of Eq. (19) if and only if one of the following equivalent conditions is satisfied:

$$a > f_a(D,\sigma), \tag{36}$$

$$D > f_D(a,\sigma), \tag{37}$$

$$\sigma < f_\sigma(a,D). \tag{38}$$

B. Scaling

With an arbitrary $\tau > 0$, we can rescale variables and parameters as

$$t' = \tau t, \quad x_i' = \tau^{-1/2}x_i, \tag{39}$$

$$a' = \tau^{-1}a, \quad D' = \tau^{-1}D, \tag{40}$$

$$\xi_i'(t') = \tau^{-3/2}\xi_i(t), \quad \sigma' = \tau^{-1}\sigma, \tag{41}$$

which leads to a system of Langevin equations equivalent to Eqs. (1):

$$\frac{d}{dt'}x_i' = a'x_i' - x_i'^3 - \frac{D'}{L-1}\sum_{j=1}^{L}(x_i' - x_j') + \xi_i'(t'), \tag{42}$$

for $i = 1,\ldots,L$, with

$$\langle\xi_i'(t')\xi_j'(s')\rangle = \sigma'^2\delta_{ij}\delta(t' - s'). \tag{43}$$

Hence, we have not three but only two independent parameters. In a similar way, also the general case with a coefficient of the cubic term in Eqs. (1) can be treated, ending with only two independent parameters.

We observe that $\phi(a,D,\sigma)$ given by Eq. (22) is invariant under this rescaling. This allows the following argument. We set $\tau = D$ such that $D' = 1$ and find $a_c' = f_a(1,\sigma')$. Therefore,

$$a_c = f_a(D,\sigma) = Df_a\left(1,\frac{\sigma}{D}\right). \tag{44}$$

Figure 1 shows a_c/D as a function of $(\sigma/D)^2$ in a log-log plot for different values of D as obtained by numerical solution of

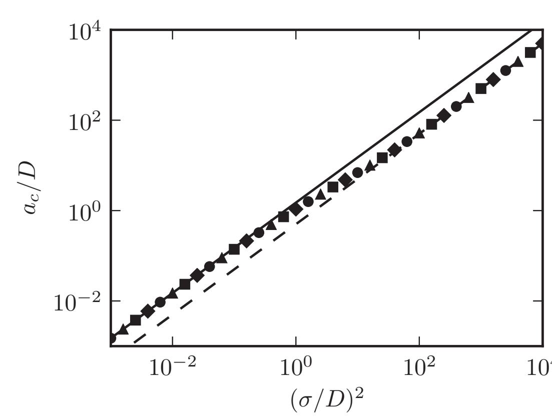

FIG. 1. Critical parameter a_c/D in dependence of noise strength $(\sigma/D)^2$ as a log-log plot. Here, Eq. (21) was numerically solved for $D = 10^{-3}$ (solid triangle), $D = 10^{-2}$ (solid circle), $D = 10^{-1}$ (solid diamond), and $D = 1$ (solid square), each for several values of σ. The solid line describes the function $3\sigma^2/(2D^2)$, the limit of weak noise, the dashed line the function $\sigma^2/(2D^2)$, the asymptote for strong noise.

Eq. (21). It confirms that a_c/D depends only on the ratio of σ and D as predicted by Eq. (44). Therefore, $a_c(\sigma,D)$ exhibits the same asymptotic behavior for strong coupling as for weak noise, and for weak coupling as for strong noise, respectively.

Dawson [7] uses a scaling where $a' = 1$ and claims that, given D', the critical noise strength σ_c' is bounded by $2^{-1/2} < \sigma_c'/D' < 2^{1/2}$ [cf. Eq. (3.43) there]. We show that this assertion is not true. From Eqs. (40) and (41), we have $\sigma_c'/D' = \sigma_c/D = f_\sigma(a,D)/D$, where $f_\sigma(a,D)$ is defined by Eq. (35). We now choose $D = 1$ and $a = f_a(\sigma = \Sigma, D = 1)$, where Σ can have any positive value and f_a is defined by Eq. (28). Since $f_\sigma(f_a(\Sigma,1),1) = \Sigma$, it holds that $\sigma_c'/D' = \Sigma$ which can be chosen beyond the bonds asserted in [7]. However, the property that for any $D' > 0$ there exists a unique critical σ_c' remains true, as we have shown in Sec. III A.

At $a = a_c(D,\sigma)$, the stationary probability density $p_s(x)$ for the coordinate of an arbitrary constituent is qualitatively different for weak and for strong noise (cf. Fig. 2). For weak noise, $p_s(x)$ is approximately a Gaussian centered at $x = 0$, whereas for strong noise it is the sum of two equally weighted narrow peaks located at $\pm\sigma\sqrt{1/D}$. For $a > a_c$, the symmetry is broken in different ways. In the weak noise limit, $p_s(x)$ is still a Gaussian but centered at $\langle x\rangle \neq 0$, whereas in the strong noise limit the two narrow peaks stay at $\pm\sigma\sqrt{1/D}$ but their weights become unequal such that $\langle x\rangle \neq 0$ also in this case. The critical parameter $a_c(D,\sigma)$ is bounded between its limit values for strong and weak noise, respectively, as will be shown rigorously in Appendix B.

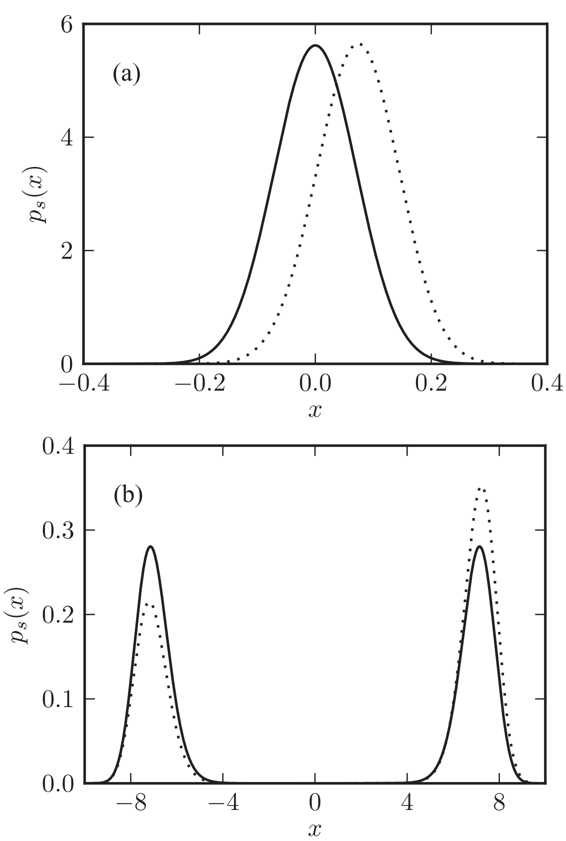

FIG. 2. Mechanism of symmetry breaking for weak and strong noise. Stationary distribution $p_s(x)$ at the critical point (solid line) and just above (dotted line). (a) For weak noise ($\sigma = 0.1$), the distribution is Gaussian, centered around zero ($m = 0$) for $a = a_c = 0.0149$, and at $a = 0.02$ rigidly shifted to the right ($m = 0.073$). (b) For strong noise ($\sigma = 10$), the distribution is bimodal with sharp peaks, centered around zero, with equal weights for $a = a_c = 52.04$ and with different weights for $a = 53$ ($m = 1.71$). Coupling strength $D = 1$.

Exploiting that at criticality the even moments are explicitly known, we give a simple hand-waving argument which leads to the correct leading behavior for weak and strong noise. Especially, we use Eq. (21), that is, $\langle x^2\rangle_{\text{crit}} = \sigma^2/(2D)$ and (cf. Appendix A) $\langle x^4\rangle_{\text{crit}} = a_c\langle x^2\rangle_{\text{crit}}$. The fourth cumulant, the kurtosis, of a Gaussian is zero and therefore $\langle x^4\rangle = 3\langle x^2\rangle^2$. Comparing this with the above expressions, we obtain for weak noise in leading order $a_c = 3\sigma^2/(2D)$, which is the upper bound in the inequality (27). Furthermore, for a symmetric probability density of two narrow peaks, the variance of x^2 is approximately zero and therefore $\langle x^4\rangle = \langle x^2\rangle^2$. Comparing with the above expressions, we obtain for strong noise in leading order $a_c = \sigma^2/(2D)$, which is the lower bound for a_c in (27).

In Eqs. (30) and (33), we have ordinary differential equations for a_c as a function of D and $1/\sigma^2$. By substituting $\alpha = a_c/D$ and $\beta = \sigma^2/D^2$, we find from either of these equations

$$\frac{d\alpha}{d\beta} = \frac{\alpha}{2\beta} + \frac{1}{2\beta}\,\frac{3/2 - \alpha/\beta}{\alpha/\beta - 1/2}. \tag{45}$$

An initial condition can be obtained by evaluating the phase transition condition (21) for an arbitrary β. We will demonstrate in Sec. III E that it is possible to obtain an analytic expression for β_0 which leads to $\alpha(\beta_0) = 1$ (see the following). Hence, this special point can serve as an initial condition and the corresponding initial value problem has a unique solution. Alternatively, we can use the limits $\alpha \to 0$ as $\beta \to 0$ or $\alpha/\beta^2 \to 0$ as $\beta \to \infty$ as initial conditions. They follow from the inequality (27). It is not clear if the initial value problem given by these limits has a unique solution. At least, the existence of a solution is guaranteed since the solution for the initial value $\alpha(\beta_0) = 1$ satisfies these limits.

In the next two sections, we will systematically study the behavior of the critical parameter for weak and strong noise using Eq. (45). The same results can be obtained by asymptotic evaluation of the integrals in the phase transition condition (21) (see Supplemental Material [20]).

C. Weak noise

Because of inequality (27), it holds that

$$a_c \to 0 \quad \text{for} \quad \sigma^2 \to 0. \tag{46}$$

That means we can continuously extend a_c to $\sigma = 0$. To obtain the asymptotic behavior of a_c for weak noise, we make the ansatz

$$a_c(\sigma,D)/D = \alpha = \sum_{n=1}^{\infty} c_n\beta^n. \tag{47}$$

Inserting this series in the differential equation (45) and comparing coefficients in powers of β, we find the recursion

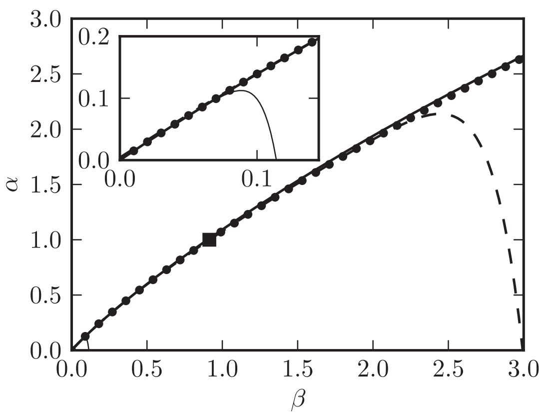

FIG. 3. Critical parameter α as a function of β. The point with $\alpha = 1$ is known exactly (solid square). The figure shows the numerical solution (solid circle) of Eq. (21), the series for weak noise up to β^{10} (thin solid line, see also insert), the corresponding Padé approximant $p_{10,10}$ (thick solid line), and the series around the exactly known point (solid square) up to 10th order (dashed line).

relation

$$c_{n+1} = \left(n - \frac{1}{2}\right)c_n - \sum_{k=1}^{n}[2(n-k)+1]c_{n-k+1}c_k \tag{48}$$

for $n = 1,2,\ldots$ with initial condition $c_1 = 3/2$. For α, the three leading terms as $\beta \to 0$ are

$$\alpha = \tfrac{3}{2}\beta - \tfrac{3}{2}\beta^2 + \tfrac{27}{4}\beta^3 + O(\beta^4). \tag{49}$$

We observe that the upper bound in (27) is reached asymptotically.

In Fig. 3, we see α as a function of β. The series (47) up to β^{10} coincides with the numerical solution of Eq. (21) only for very small values of β and taking into account more terms does not seem to improve the result for larger values of β. The figure shows also the Padé approximant $p_{10,10}$ which coincides much better with the numerical results. The Padé approximant $p_{N,N}$ is a rational function q_1/q_2, where q_1 and q_2 are polynomials of degree N and the Taylor series of $p_{N,N}$ agrees with the series (47) up to β^N [22].

We have not been able to prove convergence of the series (47) near $\beta = 0$. It is clear by its definition via the implicit function that $a_c(\sigma,D)$ is an analytic function for any positive σ, but at $\sigma = 0$ we do not know. Nevertheless, Eq. (49) has a meaning as it correctly describes the asymptotic behavior of α obtained by numerically solving the PTC (21) for $\beta \to 0$ as $\alpha \sim 3/2\,\beta$ where the symbol $\sim$ means $\lim_{\beta\to 0}\alpha/\beta = 3/2$ (cf. Fig. 1). The coefficients of the higher order terms in Eq. (49) give systematic corrections in the sense that $\alpha_1 := \alpha - 3/2\,\beta \sim -3/2\,\beta^2$ and $\alpha_2 := \alpha_1 + 3/2\,\beta^2 \sim 27/4\,\beta^3$ (cf. Fig. 4).

D. Strong noise

In the limit of strong noise $\sigma \to \infty$, again motivated by (27), we use a series ansatz for α as $\sigma^2 \to \infty$:

$$a_c/D = \alpha = c_1\beta + \sum_{i=0}^{\infty} c_{-i}\beta^{-i}. \tag{50}$$

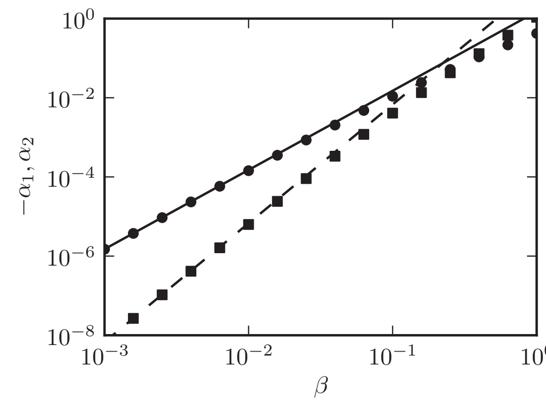

FIG. 4. Test of the scaling of a_c for small β. The symbols show the results for (solid circle) and (solid square) obtained from the numerical solution of Eq. (21). The predicted behavior $-\alpha_1 \sim 3/2\beta^2$ (solid line) and $\alpha_2 \sim 27/4\beta^3$ (dashed line) is confirmed.

Inserting in Eq. (45) and comparing coefficients in powers of β leads to

$$c_1 = \tfrac{1}{2}, \quad c_0 = 2, \tag{51}$$

and to the backward recursion

$$c_{-(n+1)} = -2c_{-n} + 4\sum_{k=0}^{n}\left(n - k + \frac{1}{2}\right)c_{-k}c_{-(n-k)} \tag{52}$$

for $n = 0,1,2,\ldots$ with initial condition $c_0 = 2$. The three leading terms for α as $\beta \to \infty$ are

$$\alpha = \frac{1}{2}\beta + 2 + 4\frac{1}{\beta} + O\left(\frac{1}{\beta^2}\right). \tag{53}$$

In the case of strong noise, we reach the lower bound in (27) asymptotically. In Fig. 5, we see α/β as a function of $1/\beta$, where the numerical solution of Eq. (21) is compared with the series (50). The series agrees with the numerical results only for very small values of $1/\beta$. Here, also the Padé approximants do not work as well as for weak noise since they have many poles within the region of interest.

As for weak noise, we have not been able to prove convergence of the series (50). Nevertheless, Eq. (53) describes

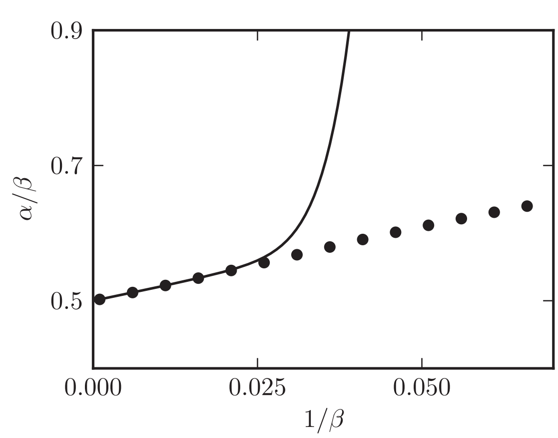

FIG. 5. Critical parameter α/β as a function of $1/\beta$. The figure shows the numerical solution (solid circle) of Eq. (21) compared with the series for strong noise (solid line) up to $(1/\beta)^{10}$.

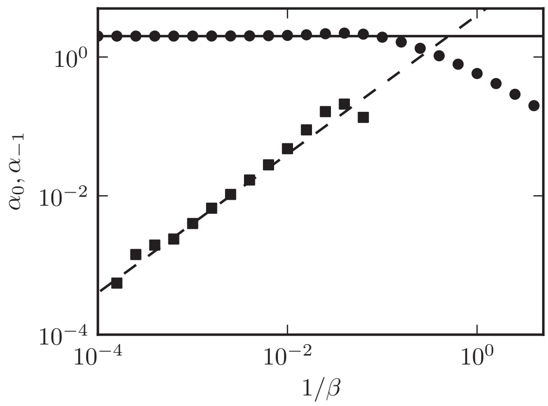

FIG. 6. Test of the scaling of a_c for large β. The symbols show the results for (solid circle) and (solid square) obtained from the numerical solution of Eq. (21). The predicted behavior $\alpha_0 \sim 2$ (solid line) and $\alpha_{-1} = 4/\beta$ (dashed line) is confirmed.

the asymptotics of α for $\beta \to \infty$ as $\alpha \sim 1/2\beta$ (cf. Fig. 1), and the higher order coefficients in Eq. (53) give systematic corrections in the sense $\alpha_0 := \alpha - 1/2\beta \sim 2$, and $\alpha_{-1} := \alpha_0 - 2 \sim 4/\beta$ (cf. Fig. 6).

E. An intermediate regime

Dawson [7] observed that, in our notation, considering only the submanifold of the critical manifold defined by the condition $a = D$, it is possible to obtain an explicit expression for the critical value of a as a function of σ. Substituting $y = x/\sqrt{\sqrt{2}\sigma}$ in Eq. (22) and solving Eq. (21) for a under the restriction that $a = D$ yields the critical parameter

$$a_c(D = a,\sigma) = 2^{-3/2}\frac{\int_{-\infty}^{\infty} dy\ \exp(-y^4)}{\int_{-\infty}^{\infty} dy\ y^2 \exp(-y^4)}\sigma = 2^{-3/2}\frac{\Gamma(1/4)}{\Gamma(3/4)}\sigma \approx 1.046\sigma. \quad (54)$$

In terms of the rescaled parameters $\alpha = a_c/D$ and $\beta = \sigma^2/D^2$, this reads as

$$\alpha\left(\beta_0 = \frac{8\Gamma(3/4)^2}{\Gamma(1/4)^2}\right) = 1. \quad (55)$$

Using the ansatz

$$\alpha(\beta) = 1 + \sum_{i=1}^{\infty} c_i(\beta - \beta_0)^i \quad (56)$$

we find with the ODE (45) the recursion formula

$$c_{n+1} = \frac{1}{(n+1)(\beta_0^2/2 - \beta_0)}\Bigg\{c_n(n + \beta_0 c_1 - 1/2 - \beta_0 n + 1/4\beta_0) + c_{n-1}(-1/2n + 3/4) + \sum_{k=1}^{n-1}[(n-k-1/2)c_k c_{n-k} + \beta_0 c_k c_{n-k+1}(n-k+1)]\Bigg\} \quad (57)$$

for $n = 2,3,\ldots$ with initial conditions

$$c_1 = \frac{1}{2-\beta_0}, \quad c_2 = \frac{1 - \beta_0 - \beta_0^2/4}{\beta_0(2-\beta_0)^3}. \quad (58)$$

In Fig. 3, we see good agreement between the numerical solution of the phase transition condition (21) and the series (56) up to the 10th order term.

IV. CRITICAL BEHAVIOR

A. Order parameter

To calculate the behavior of the order parameter m for a close to the critical value a_c, it is convenient to introduce the notation

$$N_k(m,a,D,\sigma) := \int_{-\infty}^{\infty} dx\, x^k \exp\left[\frac{2}{\sigma^2}\left(mDx + \frac{a-D}{2}x^2 - \frac{1}{4}x^4\right)\right]. \quad (59)$$

$F(m)$ defined in Eq. (17) can be expressed as

$$F(m) = \frac{N_1}{N_0} = \frac{\sigma^2}{2D}\partial_m \ln N_0(m,a,D,\sigma) \quad (60)$$

and the kth moment of the probability density $p_s(x,m)$ as

$$\langle x^k \rangle = \frac{N_k(m,a,D,\sigma)}{N_0(m,a,D,\sigma)}. \quad (61)$$

Expanding the right-hand side of Eq. (60) for small m, according to (19) we obtain the self-consistency equation

$$m = \left(\frac{2D}{\sigma^2}\right)\langle x^2\rangle_0\ m + \left(\frac{2D}{\sigma^2}\right)^3\left(\frac{\langle x^4\rangle_0}{6} - \frac{\langle x^2\rangle_0^2}{2}\right)m^3 + O(m^5), \quad (62)$$

where $\langle x^k\rangle_0 = \langle x^k\rangle|_{m=0}$.

Equation (62) has always the trivial solution $m = 0$. For $a > a_c$, there are a pair of nontrivial real solutions

$$m_\pm = \pm\frac{\sigma^2}{2D}\sqrt{\frac{\langle x^2\rangle_0 - \frac{\sigma^2}{2D}}{\langle x^2\rangle_0^2/2 - \langle x^4\rangle_0/6}} \quad (63)$$

since the denominator of the radicand is always positive as proven in Appendix B and the numerator of the radicand is positive if and only if $a > a_c$. This follows from the monotonicity of the second moment as a function of a [cf. inequality (26)]. Hence, the expansion (62) is sufficient to determine the leading behavior of m close to the critical point and we can exclude the existence of a tricritical point.

We now expand the right-hand side of Eq. (63) for small $a - a_c = \varepsilon > 0$, exploiting that at $a = a_c$ all even moments can be determined recursively from Eq. (23) (see Appendix A). Inserting Eqs. (23), (A4), and (A6) in Eq. (63) yields in leading order

$$m_\pm = \pm\frac{\sqrt{6}\sigma}{2D}\sqrt{\frac{a_c - \frac{\sigma^2}{2D}}{\frac{3\sigma^2}{2D} - a_c}}\,\varepsilon^{1/2}. \quad (64)$$

Hence, we have found the typical mean field exponent $\frac{1}{2}$ and an analytic expression for the amplitude in terms of the critical

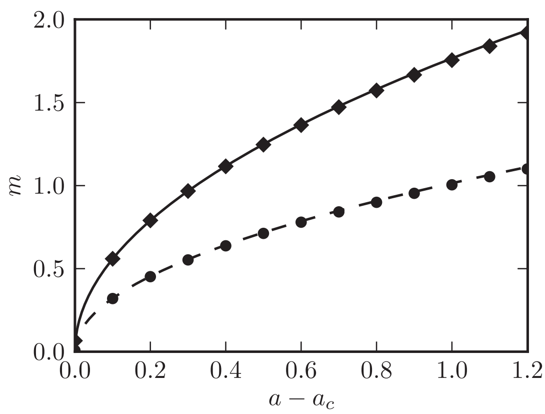

FIG. 7. Critical behavior of the order parameter m for $\sigma = 10$ (solid diamond) and $\sigma = 0.1$ (solid circle) determined by numerical evaluation of Eq. (19) compared to the analytical results from Eqs. (65) and (66) for weak noise (dashed line) and for strong noise (solid line). Coupling strength $D = 1$.

parameter. Inserting a_c in the limit of weak noise from Eq. (49) leads to

$$m_\pm = \pm\left(1 + \frac{3}{2}\frac{\sigma^2}{D^2} + O(\sigma^4)\right)\varepsilon^{1/2}. \qquad (65)$$

In the limit of strong noise, we obtain with Eq. (53)

$$m_\pm = \pm\sqrt{3}\left[1 + 2\frac{D^2}{\sigma^2} + O\left(\frac{1}{\sigma^4}\right)\right]\varepsilon^{1/2}. \qquad (66)$$

In Fig. 7, these analytical results are compared for a typical parameter setting with the numerical evaluation of the self-consistency equation (19).

B. Susceptibility

In this section, we observe that the susceptibility is diverging at the critical point as $\chi \sim A_\pm/(a - a_c)$ with the amplitudes A_+ and A_- for $a > a_c$ and $a < a_c$, respectively. We explicitly calculate the amplitudes $A_\pm$ in terms of a_c and find a universal ratio between them. The whole procedure, as well as the results, are in complete analogy to equilibrium thermodynamics. However, the calculation is explicitly possible here, and up to our knowledge it has not been done in this context before.

We introduce an external field h in Eq. (1):

$$\dot{x}_i = h + a x_i - x_i^3 - \frac{D}{L-1}\sum_{j=1}^{L}(x_i - x_j) + \xi_i(t). \qquad (67)$$

The susceptibility is defined as the response of the system to a small external field

$$\chi = \left.\frac{\partial m}{\partial h}\right|_{h=0}. \qquad (68)$$

We investigate the susceptibility close to the critical point. In analogy to Eq. (62), we find the self-consistency equation for small m and h:

$$m = \left(\frac{2D}{\sigma^2}\right)\langle x^2\rangle_0(m+h) + \left(\frac{2D}{\sigma^2}\right)^3\left(\frac{\langle x^4\rangle_0}{6} - \frac{\langle x^2\rangle_0^2}{2}\right) \times (m+h)^3. \qquad (69)$$

Taking the derivative with respect to h at $h = 0$, we obtain

$$\chi = (\chi + 1)\frac{2D}{\sigma^2}\left(\langle x^2\rangle_0 + m^2\frac{2D^2}{\sigma^4}\left(\langle x^4\rangle_0 - 3\langle x^2\rangle_0^2\right)\right). \qquad (70)$$

At the critical point we find

$$\chi = \chi + 1, \qquad (71)$$

which can be satisfied only asymptotically by $\chi \to \pm\infty$.

Below the critical point we have $m = 0$ and $\langle x^2\rangle_0 < \sigma^2/(2D)$. Therefore, (70) becomes

$$\chi = \frac{\langle x^2\rangle_0}{\frac{\sigma^2}{2D} - \langle x^2\rangle_0}. \qquad (72)$$

For small $a - a_c = \varepsilon < 0$, we find the leading behavior of the susceptibility close to the critical point using Eq. (A6):

$$\chi = A_-\frac{1}{-\varepsilon} \quad \text{with} \quad A_- = \frac{\sigma^2}{a_c - \frac{\sigma^2}{2D}}. \qquad (73)$$

For $a - a_c = \varepsilon > 0$, we obtain in leading order with the help of Eqs. (64), (70), and (A6)

$$\chi = A_+\frac{1}{\varepsilon} \quad \text{with} \quad A_+ = \frac{1}{2}\frac{\sigma^2}{a_c - \frac{\sigma^2}{2D}}. \qquad (74)$$

Since $A_+ = A_-/2$, we have

$$\lim_{\varepsilon\to+0}\frac{\chi(a_c - \varepsilon)}{\chi(a_c + \varepsilon)} = 2, \qquad (75)$$

which is universal, i.e., not depending on parameters.

V. CONCLUSION

In this paper, we have proved an upper and a lower bound for the critical parameter a_c. These bounds are optimal since they are asymptotically reached for weak and for strong noise, respectively. We found an ordinary differential equation describing the critical point a_c/D as a function of $(\sigma/D)^2$, which allows us to explicitly give a recursion formula for all coefficients of the asymptotic expansion of a_c for weak and strong noise as well as for an expansion around a special point, where a_c is known exactly.

In the limits of weak and strong noise, the mechanism of symmetry breaking is qualitatively different. For weak noise and a close to a_c, the stationary distribution of the coordinates $p_s(x)$ is a Gaussian. Below the critical point, the Gaussian is centered around zero. Above, for $a > a_c$, the Gaussian is shifted in positive or negative direction, the symmetry is broken. For strong noise and a close to a_c, $p_s(x)$ consists of two narrow peaks located symmetrically with respect to zero. The symmetry is broken such that for $a > a_c$ one of the peaks gains a larger weight than the other.

We have proved that the critical manifold is well behaved, that is, if two of the three positive parameters a, D, and σ are given there exists a unique critical value of the third. The proof hinges on the knowledge of the above mentioned boundaries of a_c. The well-behavedness of the critical manifold allows us to reduce the number of parameters and implies certain scaling properties. For example, the ratio a_c/D depends on noise strength and coupling constant only as a function of σ/D, and the limits of weak noise or strong coupling and strong noise or weak coupling are equivalent.

We further have determined the critical behavior of order parameter and susceptibility. As well known, they follow as a function of $a - a_c$ power laws with the mean field exponents. We have calculated the amplitude of the order parameter in terms of the critical parameter a_c and explicitly in the limits of weak and strong noise and found for the amplitude ratio of the susceptibilities the universal law $A_-/A_+ = 2$.

It is a natural question as to whether for systems with higher order nonlinearity similar results can be obtained. It is further desirable to study the critical manifold of a system with both additive and multiplicative noise.

ACKNOWLEDGMENTS

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APPENDIX A: MOMENTS OF *x*

We need the exact recursive relations between moments which are derived by Dawson [7] exploiting the Ito formula. To keep the paper self-contained, we rederive these relations in our notation using a different argument.

By partial integration of the right-hand side of Eq. (59), we find

$$\begin{aligned} N_k &= \int_{-\infty}^{\infty} dx\, x^k \exp\left[\frac{2}{\sigma^2}\left(Dmx + \frac{a-D}{2}x^2 - \frac{1}{4}x^4\right)\right] \\ &= \frac{1}{k+1}\frac{2}{\sigma^2}\int_{-\infty}^{\infty} dx[-(a-D)x^{k+2} + x^{k+4} - Dmx^{k+1}] \\ &\quad \times \exp\left[\frac{2}{\sigma^2}\left(Dmx + \frac{a-D}{2}x^2 - \frac{1}{4}x^4\right)\right] \end{aligned} \tag{A1}$$

for $k \in \mathbb{Z}$, $k \neq -1$. Dividing by the normalization N_0 we obtain

$$(k+1)\langle x^k\rangle = \frac{2}{\sigma^2}[-(a-D)\langle x^{k+2}\rangle + \langle x^{k+4}\rangle - Dm\langle x^{k+1}\rangle]. \tag{A2}$$

In fact, Eq. (A2) is also true for $k = -1$, then $\langle x^3\rangle = a\langle x\rangle$ [cf. Eq. (16)].

For $a \leqslant a_c$, we have $m = 0$, such that all odd moments are zero by symmetry. In that case, the recursion formula (A2) simplifies for all even moments to

$$\langle x^{2k+4}\rangle = \frac{\sigma^2}{2}(2k+1)\langle x^{2k}\rangle + (a-D)\langle x^{2k+2}\rangle \tag{A3}$$

with $k = 0,1,\ldots$.

At $a = a_c$, we have already calculated $\langle x^2\rangle|_{a_c} = \sigma^2/(2D)$ in Eq. (23) and know $\langle x^0\rangle = 1$ since $p_s(x,m)$ is normalized. Therefore, it is possible to calculate all even moments using (A3). Especially, one finds

$$\langle x^4\rangle|_{a_c} = a_c\langle x^2\rangle|_{a_c} = a_c\frac{\sigma^2}{2D}, \tag{A4}$$

$$\langle x^6\rangle|_{a_c} = a_c^2\frac{\sigma^2}{2D} - a_c\frac{\sigma^2}{2} + \frac{3\sigma^4}{4D}. \tag{A5}$$

For small $a - a_c = \varepsilon > 0$, i.e., above but close to the critical point, we obtain

$$\begin{aligned} \langle x^2\rangle &= \int_{-\infty}^{\infty} dx\, x^2 p_s(x) \\ &= \langle x^2\rangle|_{a_c} + \frac{1}{\sigma^2}(\langle x^4\rangle|_{a_c} - \langle x^2\rangle^2|_{a_c})\varepsilon + O(\varepsilon^2) \\ &= \frac{\sigma^2}{2D} + \left(\frac{a_c}{2D} - \frac{\sigma^2}{4D^2}\right)\varepsilon + O(\varepsilon^2) \end{aligned} \tag{A6}$$

using Eqs. (23) and (A4).

APPENDIX B: BOUNDS OF a_c

Since the variance of x^2 is larger than zero for any extended distribution, we have

$$\langle x^4\rangle - \langle x^2\rangle^2 > 0. \tag{B1}$$

At $a = a_c$, we obtain with Eqs. (23) and (A4)

$$\frac{\sigma^2 a_c}{2D} > \left(\frac{\sigma^2}{2D}\right)^2, \tag{B2}$$

which gives the lower bound

$$a_c > \frac{1}{2}\frac{\sigma^2}{D}. \tag{B3}$$

To obtain the upper bound, we use the inequality

$$\langle x^4\rangle|_{a_c} - 3\langle x^2\rangle^2|_{a_c} < 0, \tag{B4}$$

which states that the kurtosis of x is negative at $a = a_c$ (see following). Again, with Eqs. (23) and (A4) we find

$$\frac{\sigma^2 a_c}{2D} < 3\left(\frac{\sigma^2}{2D}\right)^2, \tag{B5}$$

which gives

$$a_c < \frac{3}{2}\frac{\sigma^2}{D}. \tag{B6}$$

To show (B4), we substitute

$$x = \sqrt{\langle x^2\rangle}\, y \tag{B7}$$

such that the new coordinate y has variance one. We denote the stationary distribution of the new coordinate by $p(y)$. The inequality (B4) in the x coordinate is equivalent to the same expression in the new coordinate y:

$$\langle y^4\rangle - 3\langle y^2\rangle^2 < 0. \tag{B8}$$

Now, we compare the distribution $p(y)$ with the Gaussian distribution with variance one, which we denote by $g(y)$. In the following, we only consider the critical point $a = a_c$ where both distributions have zero mean and are symmetric under the transformation $y \to -y$. We look at the intersection of both curves and distinguish two cases. There can be either two or four intersecting points.

For two intersection points we use the following theorem [23]: If a symmetric zero-mean probability distribution $p(y)$ intersects with the standard normal distribution $g(y)$ in exactly two points $-y_0, y_0$, then $g(y) > p(y)$ for all $y > y_0$ if and only if the kurtosis of $p(y)$ is negative.

In the present situation $p(y)$, decays as $\exp(-\lambda y^4)$, $\lambda > 0$ as $y \to \pm\infty$, and therefore $g(y) > p(y)$ for large enough $|y|$. Hence, we can apply the theorem and (B8) is satisfied.

In the case of four intersection points of $p(y)$ and $g(y)$, we use the following theorem from [24]: Suppose two probability densities $g(y)$ and $p(y)$ with zero mean and the same variance are given. Let μ_{g3},μ_{g4}; μ_{p3},μ_{p4} be their respective third and fourth moments. Then, we have a sufficient condition for $\mu_{g4} \geqslant \mu_{p4}$: there should exist four abscissas $a_1 < a_2 < a_3 < a_4$ such that

$$\text{(i)} \quad \text{when} \quad \left.\begin{aligned} -\infty < y < a_1 \\ a_2 < y < a_3 \\ a_4 < y < \infty \end{aligned}\right\} \; g(y) \geqslant p(y), \tag{B9}$$

$$\text{(ii)} \quad \text{when} \quad \left.\begin{aligned} a_1 < y < a_2 \\ a_3 < y < a_4 \end{aligned}\right\} \; g(y) \leqslant p(y), \tag{B10}$$

and (iii) $a_1 + a_2 + a_3 + a_4$ and $\mu_{p3} - \mu_{g3}$ are not both strictly positive or both strictly negative.

In the present case, we have $\mu_{g3} = \mu_{p3} = 0$. Furthermore, $a_1 = -a_4$ and $a_2 = -a_3$ since both $g(y)$ and $p(y)$ are even functions. Hence, $a_1 + a_2 + a_3 + a_4 = 0$ and we can apply the theorem. Therefore, $\mu_{p4} \leqslant \mu_{g4}$ [25]. We can follow the lines in [24] to prove even strict inequality.

Consider the function $h(y) = (a_1 - y)(a_2 - y)(a_3 - y)(a_4 - y)$. For any $y \in \mathbb{R}$, the functions $g(y) - p(y)$ and $h(y)$ have either the same sign or at least one of them is zero. Thus, $h(y)[g(y) - p(y)] \geqslant 0$. In the present situation, since both functions are continuous and zero only if $y \in \{a_1,a_2,a_3,a_4\}$, there exists $\varepsilon,\delta > 0$ such that $h(y)[g(y) - p(y)] > \varepsilon$ for $y \in \mathbb{R}$, $y \notin [a_i - \delta, a_i + \delta]$ for $i = 1,2,3,4$. Hence, we have

$$\int_{-\infty}^{\infty} dy\, h(y)[g(y) - p(y)] > 0. \tag{B11}$$

Expanding the polynomial $h(y)$ and performing the integral in (B11), we find

$$\mu_{g4} - \mu_{p4} > 0, \tag{B12}$$

where we used that odd moments of $p(y)$ and $g(y)$ are zero and that both distributions have variance one and are normalized. Since the kurtosis of a Gaussian is zero, by (B12) we follow that (B8) is true, which completes the proof of the inequality (27).

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[25] Dawson [7] already showed that the fourth cumulant is less or equal to zero [cf. Eq. (4.15) there]. By numerical evidence, he conjectured that there is strict inequality. We prove this conjecture.